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The Decay of a Stellar Magnetic Field

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THE DECAY OF A STELLAR MAGNETIC FIELD

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Abstract

The decay of the solar magnetic field may be computed from a variational principle based on Lenz's law: i.e., since the fields resist any changes, they adjust themselves to make the decay time a maximum. A variational calculation of the decay time associated with one mode of the solar magnetic field has been made and compared with an exact value obtained by Wrubel. To the accuracy of the calculation, a simple, practically constant, trial function, plus one iteration, reproduces the exact result identically.

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1. Introduction

The decay time associated with the solar magnetic field was investigated some years ago by Cowling[1], who used a variational method to calculate the decay time associated with the lowest mode. More recently, Wrubel[2] computed decay times for several higher modes. Although Wrubel analyzed the higher modes by integrating the differential equation of the problem numerically, Cowling's use of the variational method may be extended to apply to the higher modes also, and it is this aspect of the problem that will concern us here.

A general discussion of the use of variational techniques for the calculation of decay times was given in a previous report[3]. This included both a set of variational expressions for computing decay times, and a vector iterational procedure for systematically improving the trial functions. It was also pointed out that the stationary property of the variational principles was the mathematical expression of the physical fact that, since the actual fields resist any change, they adjust themselves to make the decay time a maximum (Lenz's law).

In the report mentioned[3], the aim was to present an exhaustive treatment of the various variational principles encountered in decay problems; the discussion was therefore kept general. Here, in contrast, we wish to discuss only the specific problem considered by Wrubel[2]. A single variational principle is used and detailed numerical results are obtained. In particular, we show that to the accuracy of the calculation, a simple (practically constant) trial function, plus one iteration, yields a decay time identical with the result obtained by Wrubel.

We note that this report, and the preceding one, may be read independently.

2. Assumptions

In order to simplify the calculation, we shall make certain assumptions concerning a) magnitude, and b) mathematical form. In this section we discuss our assumptions, their justification, and the simplifications that result.

If c is the velocity of light, τ the decay time, σ the conductivity (e.s.u.), and L a typical dimension of the conducting region,

$$\tau \sim \frac{\sigma L^2}{c^2} \; . \tag{2.1}$$

We now assume that the decay times are always very large; that is, using (2.1), we require*

$$\left(\frac{c\tau}{L}\right)^2 \sim \sigma\tau \gg 1 \; . \tag{2.2}$$

The first form states that during the decay time an electromagnetic disturbance can travel a distance much larger than L; the second form states two equivalent facts: a) the conduction current is much larger than the displacement current, and b) the volume charge decays in a time very much less than the decay time.

In Gaussian units, with $\varepsilon = \mu = 1$, the magnetic field is always divergenceless; because the volume charge disappears so quickly, the electric field may also be regarded as divergenceless. Thus,

$$\nabla \cdot \underline{H} = 0 \; , \tag{2.3}$$

$$\nabla \cdot \underline{E} = 0 \; . \tag{2.4}$$

Since the fields decay as $e^{-t/\tau}$, the first Maxwell equation becomes:

$$\nabla \times \underline{E} = \frac{\underline{H}}{c\tau} \; . \tag{2.5}$$

---

* In our problem, using the sun's radius for L, $\left(\frac{c\tau}{L}\right)^2 \sim 10^{32}$. At the center of the sun, $\sigma\tau \sim 10^{34}$; this falls to 10^{31} on the shell having a radius 90% of that of the sun.

In the second Maxwell equation, we omit the displacement current:

(2.6) $$\nabla \times \underline{H} = \frac{4\pi}{c} \sigma \underline{E} .$$

The neglect of the displacement current, $\frac{\underline{E}}{c\tau}$, in (2.6) is justified by (2.2): according to (2.5), $\underline{E} \sim \underline{H} \frac{L}{c\tau}$; therefore, in (2.6), the displacement current is of order $\left(\frac{L}{c\tau}\right)^2$ compared with the curl $\underline{H}$ term, and, if the restriction (2.2) applies, may be dropped. The theory resulting from Eqs. (2.5) and (2.6) is sometimes called the 'quasi-steady' theory.

Eqs. (2.3) and (2.5) are consistent. Comparing Eqs. (2.4) and (2.6) leads to an additional relation:

(2.7) $$\underline{E} \cdot \nabla \sigma = 0 .$$

In our problem, σ is spherically symmetric; (2.7) therefore requires that $\underline{E}$ have no radial component (H-modes only).

Outside the conducting region (radius R, surface S, volume V) the above equations (with $\sigma = 0$) still apply, at least up to distances L that satisfy (2.2); beyond these distances, the field is negligibly small. That is, outside the conducting sphere both $\underline{E}$ and $\underline{H}$ satisfy Laplace's equation, with the two fields coupled according to (2.5).

The $\underline{E}$ field in the external region may be replaced, in its influence on the $\underline{E}$ field in the conducting region, by a boundary condition. We express the boundary condition in terms of the unit normal $\underline{n}$ (pointing out of the conductor) and the (symmetric) dyadic y':

(2.8) $$(\nabla \times \underline{E}) \times \underline{n} = y' \cdot \underline{E} \qquad \text{(on S).}$$

The notation has been chosen to agree with that used previously[3].

Eliminating $\underline{H}$ from (2.5) and (2.6),

$$\left(\nabla^2 + \frac{4\pi\sigma}{c^2\tau}\right) \underline{E} = 0, \tag{2.9}$$

where the Laplacian operates on the rectangular components of $\underline{E}$. Eqs. (2.4), (2.7), (2.8) and (2.9) constitute a vector eigenvalue problem. We now reduce it to a scalar problem by specializing the form of $\underline{E}$.

Introducing a spherical coordinate system (r,θ,ϕ) with unit basis vectors $\underline{e}_r$, $\underline{e}_\theta$, $\underline{e}_\phi$, we limit the electric fields to the forms considered by Wrubel[2], namely,

$$\underline{E} = \frac{p(r)}{r} \quad P_n^1(\mu)\, \underline{e}_\phi \;. \tag{2.10}$$

Here, $\mu = \cos\theta$, and P_n^1 is the associated Legendre function of degree n and order 1.

Fields of the form (2.10), since they are independent of ϕ, satisfy (2.4); since they have no radial component, they also satisfy (2.7). All that remains is to satisfy Eqs. (2.8) and (2.9).

Consider (2.9). In terms of the rectangular unit vectors $\underline{e}_x$, $\underline{e}_y$, $\underline{e}_z$, (2.10) becomes:

$$\underline{E} = \frac{p(r)}{r} \operatorname{Re}\left\{ Y_n^1(\theta,\phi)(\underline{e}_y + i\,\underline{e}_x)\right\}, \tag{2.11}$$

where Re means 'real part of' and Y_n^1 is the spherical harmonic of degree n and order 1.

Using (2.11) in (2.9):

$$\frac{d^2p(r)}{dr^2} - \frac{n(n+1)}{r^2}\, p(r) + \frac{4\pi}{c^2\tau}\,\sigma(r)p(r) = 0. \tag{2.12}$$

Now we turn to (2.8). Outside the conductor, $\sigma = 0$ in (2.12). Therefore,

$$p(r) \sim \frac{1}{r^n} \; ; \qquad r > R . \tag{2.13}$$

Putting (2.10) in (2.8), the boundary condition reduces to:

$$\frac{1}{p}\,\frac{dp}{dr} = \underline{e}_{\phi} \cdot y' \cdot \underline{e}_{\phi} = y'_{\phi\phi} \; ; \qquad r = R . \tag{2.14}$$

$y'_{\phi\phi}$ is evaluated from (2.13); (2.14) then becomes:

$$\frac{r}{p}\,\frac{dp}{dr} + n = 0 \quad ; \qquad r = R . \tag{2.15}$$

The electric field must be finite at the origin; therefore,

$$p(0) = 0 . \tag{2.16}$$

We now have a scalar eigenvalue problem: namely, to find those values of τ for which there exists a solution of (2.12) subject to the boundary conditions (2.15) and (2.16).

Wrubel[2] solved this problem by assuming a value for τ in (2.12) and integrating (2.12) numerically from $r = 0$ to $r = R$; p and $\frac{dp}{dr}$ at $r = R$ were then tested by comparison with (2.15); if (2.15) was not satisfied, the process was repeated with a new value assumed for τ until, eventually, all conditions of the problem were satisfied.

3. Variational-iterational procedure[†]

In this section we shall set up a variational iterational procedure for solving the scalar eigenvalue problem presented by Eqs. (2.12), (2.15), and (2.16) for $n = 1$.

[†] The general method is given by G. Temple and W.G. Bickley[4]. The application to magnetic decay problems is discussed in Ref. [3]. Several variational principles are given in Ref. [3]; in particular, a (vector) variational iterational procedure for computing τ from the electric field distribution alone is described; if (2.10) is substituted in this variational-iterational scheme, the scalar equations of this section follow.

First consider the variational principle:

$$(3.1) \qquad \frac{4\pi}{c^2\tau} = \frac{\int_0^R dr \left\{ \frac{2p^2}{r^2} - p\,\frac{d^2p}{dr^2} \right\}}{\int_0^R dr\ \sigma\ p^2} \quad .$$

In (3.1), $\delta\tau = 0$ for variations of p about the solution of (2.12) subject to the restrictions (2.15) and (2.16)[3]† (putting $n = 1$).

The spectrum of τ is discrete and bounded from above[3]. We limit ourselves to the largest eigenvalue for $n = 1$ and call this eigenvalue $\tau_{1,0}$ (no node). The value of τ obtained by inserting a trial field[**] in (3.1) will be $\leq \tau_{1,0}$.

Call this initial trial field p_1. A sequence of iterated functions p_k $(k = 2,3,\ldots)$ is defined by those solutions of

$$(3.2) \qquad \frac{d^2p_n}{dr^2} - \frac{2}{r^2}\,p_k = -\frac{4\pi}{c^2}\,\sigma\,p_{k-1}$$

that satisfy

$$(3.3) \qquad p_k(0) = 0$$

$$(3.4) \qquad \frac{r}{p_k}\,\frac{dp_k}{dr} = -1 \quad ; \qquad r = R \, .$$

An explicit solution of these equations is provided by the Green's function

$$(3.5) \qquad \begin{aligned} G(r,r') &= \frac{r^2}{3r'} \quad ; \qquad && r < r' \\ &= \frac{r'^2}{3r} \quad ; \qquad && r > r' \, . \end{aligned}$$

[**] Assumed not orthogonal to the eigenfunction belonging to $\tau_{1,0}$.

Since $G(r,r')$ satisfies (3.3), (3.4) and

$$\frac{d^2}{dr^2} G(r,r') - \frac{2}{r^2} G(r,r') = -\delta(r-r') \tag{3.6}$$

we find

$$p_k(r) = \int_0^R dr' \, G(r,r') \frac{4\pi}{c^2} \sigma(r') \, p_{k-1}(r') \; . \tag{3.7}$$

According to (3.5), $G(r,r') = G(r',r)$. Therefore, from (3.7),

$$\int_0^R dr \, p_k(r) \, \sigma(r) \, p_{\ell-1}(r) = \int_0^R dr' \, p_{k-1}(r') \, \sigma(r') \, p_\ell(r') = (k+\ell-1), \tag{3.8}$$

where the last notation emphasizes that only the sum of the subscripts enters into the value of the integral.

We put $p = p_k$ in (3.1) and label the resulting value of the decay time τ_{2k}. Using (3.2),

$$\tau_{2k} = \frac{\int_0^R dr \, \sigma \, p_k^2}{\int_0^R dr \, \sigma \, p_k p_{k-1}} = \frac{(2k)}{(2k-1)} \; . \tag{3.9}$$

It may now be shown[3]† that, as $k \to \infty$, the τ_{2k} increase in value, approaching $\tau_{1,0}$ as a limit, while the functions p_k converge towards the exact eigenfunction corresponding to $\tau_{1,0}$.

4. Numerical results

We have already taken $n = 1$. If we introduce a dimensional variable x by

(4.1) $$x = \frac{r}{R} \; ; \qquad 0 \leq x \leq 1 \; ,$$

the integrals in (3.8) become:

(4.2) $$(k + \ell) = R\,\sigma(0) \int_0^1 dx \; \frac{\sigma(x)}{\sigma(0)} \; p_k(x)\, p_\ell(x) \; .$$

The variation of the function $\frac{\sigma(x)}{\sigma(0)}$ in the sun (in Wrubel's notation $\theta^{3/2}$) has been tabulated by Wrubel in steps of .1. The values given by Wrubel are reproduced in Table I. Since $\frac{\sigma(x)}{\sigma(0)}$ is only available numerically, the integrals of the form of (4.2) that occur have been evaluated by numerical methods. The procedure has been to apply Simpson's Rule, using as a net the 10 subdivisions appearing in Table I. Note that the values taken on by the p_k at $x = 1$ are unimportant since the p_k occur in (4.2) multiplied by $\frac{\sigma(x)}{\sigma(0)}$ and this factor vanishes at $x = 1$. The practical result of this is that the boundary condition (3.4) may be ignored in this problem.

At $x = 0$, however, the boundary condition must be included. We have done this for $p_1(x)$ by choosing this function to be:

(4.3) $$p_1(x) = \begin{cases} 0 \; ; & x = 0 \\ 1 \; ; & x \neq 0 \end{cases} \; .$$

Using (4.2) and (4.3), we find:

(4.4) $$(2) = .202\, R\, \sigma(0) \; .$$

Iterating $p_1(x)$ by use of (3.7), we find:

$$p_2(x) = 4\pi\, R^2\, \frac{\tau(0)}{c^2} \left\{ \frac{1}{3} \left[\frac{1}{x} \int_0^x dx'\, x'^2\, \frac{\sigma(x')}{\sigma(0)}\; p_1(x') + x^2 \int_x^1 dx'\, \frac{\sigma(x')}{\sigma(0)}\; \frac{1}{x'}\, p_1(x') \right] \right\}$$

(4.5)

$$= 4\pi\, R^2\, \frac{\sigma(0)}{c^2}\; \alpha_2\,(x)$$

thereby defining $\alpha_2(x)$. Values of $\alpha_2(x)$ computed from (4.3) and (4.5) are listed in Table I.

With $p_2(x)$ available, we obtain from (4.2),

(4.6) $$(3) = 1.087 \times 10^{-3}\; 4\pi\, R^2\, \frac{\sigma(0)}{c^2} \quad R\, \sigma(0)$$

(4.7) $$(4) = 6.54 \times 10^{-6} \left(4\pi\, R^2\, \frac{\sigma(0)}{c^2}\right)^2 \quad R\, \sigma(0)\,.$$

Before combining these values according to (3.9) we must have a value for the factor $4\pi R^2 \sigma(0)/c^2$. In Wrubel's work this factor ($\Gamma\tau$ in Wrubel's notation) has the value (see Wrubel's Table 3)

(4.8) $$4\pi\, R^2\, \frac{\sigma(0)}{c^2} = 6.56 \times 10^{11} \text{ years}$$

(Note that Wrubel's σ and ours differ by a factor c^2).

Collecting these results, we obtain two estimates for $\tau_{1,0}$:

(4.9) $$\tau_3 = 3.53 \times 10^9 \text{ years}$$

(4.10) $$\tau_4 = 3.96 \times 10^9 \text{ years}\quad .$$

As expected, $\tau_4 > \tau_3$. Wrubel's value for the decay time is:

(4.11) $$\tau_{\text{Wrubel}} = 4.08 \times 10^9 \text{ years} .$$

This differs by 3% from τ_4.

However, we should note that Wrubel uses a subdivision that employs 10 times the number of points we use. To evaluate the effect of our coarser net, we have used Wrubel's 'exact' values for p(x) and $\frac{dp}{dx}$ in (3.1). As in the previous evaluations, the integrals were done by Simpson's Rule using only the ten subdivisions given in Table I. Since it is this value of τ, rather than (4.11), that should be compared with the variational results τ_3, τ_4, we call this decay time τ_{exact}. We find:

(4.12) $$\tau_{\text{exact}} = 3.96 \times 10^9 \text{ years} .$$

Thus, the 3% difference between (4.11) and (4.10) may be considered as arising solely from the fact that the net used for (4.11) was 10 times finer than that used for (4.10).

We conclude that the use of the simple trial function (4.3), plus one iteration, in the variational principle duplicates, to the accuracy of the calculation, the value Wrubel obtained by numerical integration of the differential equation.

TABLE I

x	$\frac{\sigma(x)}{\sigma(0)}$	$a_2(x)$
0.0	1.000	0.00
.1	.808	3.99×10^{-3}
.2	.474	6.55×10^{-3}
.3	.264	7.60×10^{-3}
.4	.146	7.70×10^{-3}
.5	.0816	7.34×10^{-3}
.6	.0447	6.74×10^{-3}
.7	.0231	6.07×10^{-3}
.8	.0103	5.44×10^{-3}
.9	.00305	4.64×10^{-3}
1.0	.00000	4.36×10^{-3}

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[1] Cowling, T.G. - M.N., 105, 166 (1945).

[2] Wrubel, M.H. - Astrophys. J., 116, 291 (1952).

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